

MODAL ANALYSIS OF PLANE FRAMES

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MODAL ANALYSIS OF PLANE FRAMES

Project Report Submitted in partial fulfillment of the requirements for the degree of

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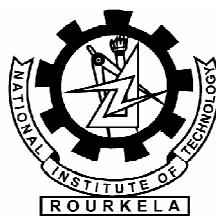
Civil Engineering

by

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CERTIFICATE

This is to certify that the project entitled ***Modal Analysis of Plane Frames*** submitted by Mr. ***Ravi Gera*** (Roll No. **107CE038**) and Ms. ***Nipika Paul*** (Roll. No. **107CE020**) in partial fulfillment of the requirements for the award of Bachelor of Technology Degree in Civil Engineering at NIT Rourkela is an authentic work carried out by them under my supervision and guidance.

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ABSTRACT

In the modal analysis of plane frames the main aim is to determine the natural mode shapes and frequencies of an object or structure during free vibration. The dynamic analysis of frames requires inclusion of axial effect in the stiffness and mass matrices. It also requires a co-ordinate transformation of the nodal or local co-ordinates to global co-ordinates. The analysis is performed using Modal Analysis.

The physical interpretation of the eigenvalues and eigenvectors which come from solving the system are that they represent the frequencies and corresponding mode shapes.

The objective of this project is to study the vibration, frequency and mode shape of plane frames. Formulation of stiffness matrix and mass matrix are to be done using direct stiffness method. Programs are to be developed using ANSYS and MATLAB codes. A modal analysis of a plane frame will also be studied.

CHAPTER 1

INTRODUCTION

Modal analysis is the study of the dynamic properties of structures under vibrational excitation.[10]

When a structure undergoes an external excitation, its dynamic responses are measured and analysed. This field of measuring and analysing is called modal analysis. Modal analysis can be used to measure the response of a car's body to a vibration when vibration of a an electromagnetic shaker is attached, or the pattern created by noise because of a loudspeaker which acts as an excitation.[10]

In structural engineering, modal analysis is applied to find the various periods that the structure will naturally resonate at, by using the structure's overall mass and stiffness. The modal analysis is very important in earthquake engineering, because the periods of vibration evaluated helps in checking that a building's natural frequency does not coincide with the frequency of earthquakes prone region where the building is to be constructed. Incase a structure's natural frequency coincidently equals an earthquake's frequency, the structure suffers severe structural damage due to resonance. [10]

The frequency and mode shape of a model is determined by modal analysis. When the models are subjected to cyclic or vibration loads, the dynamic response of structures due to these external loads acting, which include resonance frequencies (natural frequencies), mode shape and damping, are estimated.

Natural Frequency: All models have a natural frequency. If a model is subjected to dynamic load that is close to its natural frequency, the model oscillates to a large extent than in normal condition. The results of a modal analysis help determine whether a model requires more or

less damping to prevent failure. Modal analysis can be used to find the frequency at which resonance occurs, under specific constraints.

Modes: Modes measure the vibration of an object at a particular frequency. Each mode is assigned a number. The lowest speed at which a structure vibrates after all external loads are removed is assigned to mode 1. This mode is called the free vibration mode because it is not damped.

Mode shape: In the study of vibration in engineering, the expected curvature (or displacement) of a surface at a particular mode due to vibration is the mode shape. To determine the vibration of a system, Multiplying the mode shape by a time-dependent function, the vibration of a system is found out. Thus the mode shape always describes the time-to-time curvature of vibration where the magnitude of the curvature will change. The mode shape depends on two factors:

- 1) on the shape of the surface
- 2) the boundary conditions of that surface.

Eigenvalues and eigen vectors: The eigen-values and eigenvectors of the stiffness matrix have some physical meanings. An eigen-value and its associated eigenvector can be interpreted as a strain energy and corresponding displacement mode, respectively. On the basis of this physical meaning, an eigenvector can be graphically visualized as a displacement mode which results in the strain energy relevant to the associated eigenvalue.[11]

The set of an eigenvalue and associated eigenvector is termed “**eigen mode**”. There are as many eigen modes as the number of free d.o.f. in the model, which may consist of a single element, a number of selected elements, or the whole system. The model may or may not include the boundary constraints, which are reflected in eigen modes. The integration rules as well as the element properties assigned to the model are also reflected in the displayed eigen modes.[11]

Shear building : To carry out the modal analysis of a multiple-degree-of-freedom, an idealized two-storey frame with forces externally acting on the structure is assumed. The members like floors and beams are infinitely rigid in flexure. Some factors like deformation of the members axially, stiffness of the members being affected by the forces acting axially, are neglected. This shear frame idealization helps in developing the equations of motion for a system with multiple-degree-of-freedom.[12]

Using the lumped mass system, the distributed mass is idealized at the levels of the floors as concentrated mass. The number of degrees of freedom of a structure is defined as the number of displacements occurring independently that defines the position by which the masses are displaced in comparison to the equilibrium position originally.[12]

CHAPTER 2

LITERATURE REVIEW

2.1 MODAL ANALYSIS OF BEAMS:

Krawczuk, Ostachowicz and Zak [5] had studied Modal analysis of composite beam that is cracked and unidirectional. With a single transverse fatigue crack, a model along with an algorithm were presented to create the characteristic matrices of a composite beam. The influence of the crack parameters (position and relative depth) and the material parameters (relative volume and fibre angle) on changes in the first four transverse natural frequencies of the composite beam made from unidirectional composite material has been analysed by applying the element that developed.

Hong Hee Yooa, Jung Eun Chob, and Jintai Chungc [4] studied Modal analysis of cantilever beams undergoing rotation and its shape and shape optimization. The modal characteristics of a cantilever beam rotating about an axis perpendicular to its neutral-axis, vary significantly. When a well-established analysis procedure is followed by applying assumed mode method or finite element method, the variation in the modal characteristics can be estimated accurately, when the geometric shape and material property of a beam are given. When some modal characteristics are specified as design requirements in practical design situations, the geometric shape that satisfies the requirements are estimated. In this study, the geometric shapes that satisfy the modal characteristic requirements such as maximum or minimum slope natural frequency loci, are obtained through an optimization procedure.

Maurinia, , M. Porfirib, and J. Pougeta [3] studied ‘Numerical methods for modal analysis of stepped piezoelectric beams’ . Stepped piezoelectric beams that were modeled using the

Euler–Bernoulli beam theory, were used to analyze different numerical methods for modal analysis. The results from standard numerical approaches based on the discretization of the stepped beam (assumed modes and finite-element methods) and the solution of the exact transcendental eigenvalue problem for the infinite dimensional system were compared.

Carlos E.N. Mazzillia, César T. Sanches, Odulpho G.P. Baracho Neto, Marian Wiercigroch and Marko Keber [2] studied ‘Non-linear modal analysis for beams subjected to axial loads’. To build-up robust low-dimensional models, non-linear equations obeying the laws of dynamics of an axially loaded beam were derived. A structure was subjected to two important loading conditions, a uniformly distributed axial and a thrust force. By developing and applying methodological analysis, these loads were made to imitate the main forces acting on an offshore riser. Method of multiple scales was used to construct non-linear normal modes (NNMs) and non-linear multi-modes (NMMs) to analyse the transversal vibration responses by monitoring the modal responses and mode interactions. The developed analytical models and the results from FEM simulation were verified. The FEM model having 26 elements and 77 degrees-of-freedom was checked with the results of the low-dimensional (one degree-of-freedom) non-linear oscillator developed by constructing a so-called invariant manifold. The dynamical responses were compared in terms of time histories, phase portraits and mode shapes.

2.2 MODAL ANALYSIS OF TRUSS

Wei Gao [7] studied ‘Interval natural frequency and mode shape analysis for truss structures with interval parameters’. In this paper, Euler–Bernoulli beam theory was applied for modeling of stepped piezoelectric beams to analyse different numerical methods for obtaining the modal analysis of the beam. Results from standard numerical approaches that

relied on the discretization of the stepped beam (assumed modes and finite-element methods), were compared with the solution of the exact transcendental eigenvalue problem for the infinite dimensional system. An accurate and manageable novel method was obtained that improved the assumed modes basis functions with special jump functions. Numerical results were compared with experimental data and the adopted beam model was checked for accuracy and then validated.

2.3 MODAL ANALYSIS OF PLANE FRAMES:

Y. L. Xu, and W. S. Zhang [8] studied ‘Modal analysis and seismic response of steel frames with connection dampers’. In between the connection of end plate and column flange or between the angle and member flange, dissipation materials may be placed in order to minimize dynamic response of a steel frame of bolted connections. He derived the mass matrix, stiffness matrix and damping matrix for the frame using a combination of the finite element method and the direct stiffness method to idealize bolted connections and energy dissipation materials as rotational spring and damper respectively. Using the complex modal analysis, dynamic characteristics of the frame was analysed and the effects of connection stiffness and rotational damper on natural frequency and modal-damping ratio were enquired. In order to observe the effect of connection stiffness and rotational damper on the seismic performance of the frame, a generalized pseudo-excitation method was introduced for the vibration analysis of the frame subject to earthquake excitation. The parametric studies on the example frame with and without the connection dampers showed that there was an optimal damper damping coefficient by which the modal-damping ratio of the frame can be considerably increased and the seismic responses, including both lateral displacements and internal forces, can be significantly reduced.

Siu Lai Chana [6] studied ‘Vibration and modal analysis of steel frames with semi-rigid connections’. In practical design, frames are assumed to be connected either by rigid or pinned joints. Strictly speaking, all joints are semi-rigid, so this assumption does not normally represent the actual behaviour of a realistic steel frame. The response of a steel frame will be of interest to the designer for the purpose of ensuring the safety and serviceability of the structure under the action of dynamic loads. This paper was addressed to the extension of the well known stiffness matrix method of analysis for framed structures to cover a more general case of vibration and stability analysis of flexibly connected steel frames.

A.A. Vasilopoulou and D.E. Beskos [1] studied ‘Seismic design of plane steel frames using advanced methods of analysis’. A rational and efficient seismic design methodology for plane steel frames using advanced methods of analysis in the framework of Eurocodes 8 and 3 is presented. This design methodology employs an advanced finite-element method of analysis that takes into account geometrical and material nonlinearities and member and frame imperfections. Seismic loads and resistances (strengths) are computed according to Eurocodes 8 and 3, respectively. The inelastic dynamic method in the time domain is employed with accelerograms taken from real earthquakes scaled so as to be compatible with the elastic design spectrum of Eurocode 8. The design procedure starts with assumed member sections, continues with the drifts, damage, plastic rotation and plastic hinge pattern checks for the damage, the ultimate and the serviceable limit states and ends with the adjustment of member sizes. Thus it can sufficiently capture the limit states of displacements, strength, stability and damage of the structure and its individual members so that separate member capacity checks through the interaction equations of Eurocode 3 or the usage of the conservative and crude q -factor suggested in Eurocode 8 are not required. Two numerical examples dealing with the design of one bay and four storey and two bay and seven storey

moment resisting frames are presented to illustrate the method and demonstrate its advantages.

2.4 OUT LINE OF PRESENT WORK

The objective of this project is to study the vibration, frequency and mode shape of plane frames. Formulation of frequency, stiffness matrix and mass matrix are to be done. Programs are developed using ANSYS and MATLAB codes.

CHAPTER 3

THEORY AND FORMULATION

3.1 FORMATION OF STIFFNESS MATRIX

3.1.1 Introduction

Rigid joints interconnect all the members lying in the same plane, in plane frames. Bending moment, shear force and an axial force are the resultants of internal stress at a cross-section of the member of a plane frame. The plane frame is subjected to flexural and axial deformations only. Direct stiffness matrix method is employed for the analysis of plane frames. [11]

Two steps are to be followed for establishing the stiffness matrix. First, the stiffness matrix of the plane frame member is derived in its local co-ordinate axes. Secondly, the stiffness matrix in the local co-ordinate axes already deduced is transformed to global co-ordinate system.

The reason for the transformation of local co-ordinate system to global co-ordinate system is that - Orientation of plane frame members are in varied directions and hence prior to the formation of the global stiffness matrix all the member stiffness matrices must be referred to the same set of axes. Thus, it is essential that all the forces and displacements are transformed to the global co-ordinate system.[11]

3.1.2 Member Stiffness Matrix

In Fig. 1a, we assume a member of a plane frame in the member

co-ordinate system $x'y'z'$ whose global orthogonal set of axes are xyz . Some of the assumptions to be considered are:

- 1) The plane in which the frame lies is x - y plane.
- 2) All the members of the plane frame should have uniform flexural rigidity EI and uniform axial rigidity EA .

This modal analysis takes the axial deformation of member into consideration. The possible displacements at each node of the member are: translation in x' - and y' - direction and rotation about z' - axis. [11]

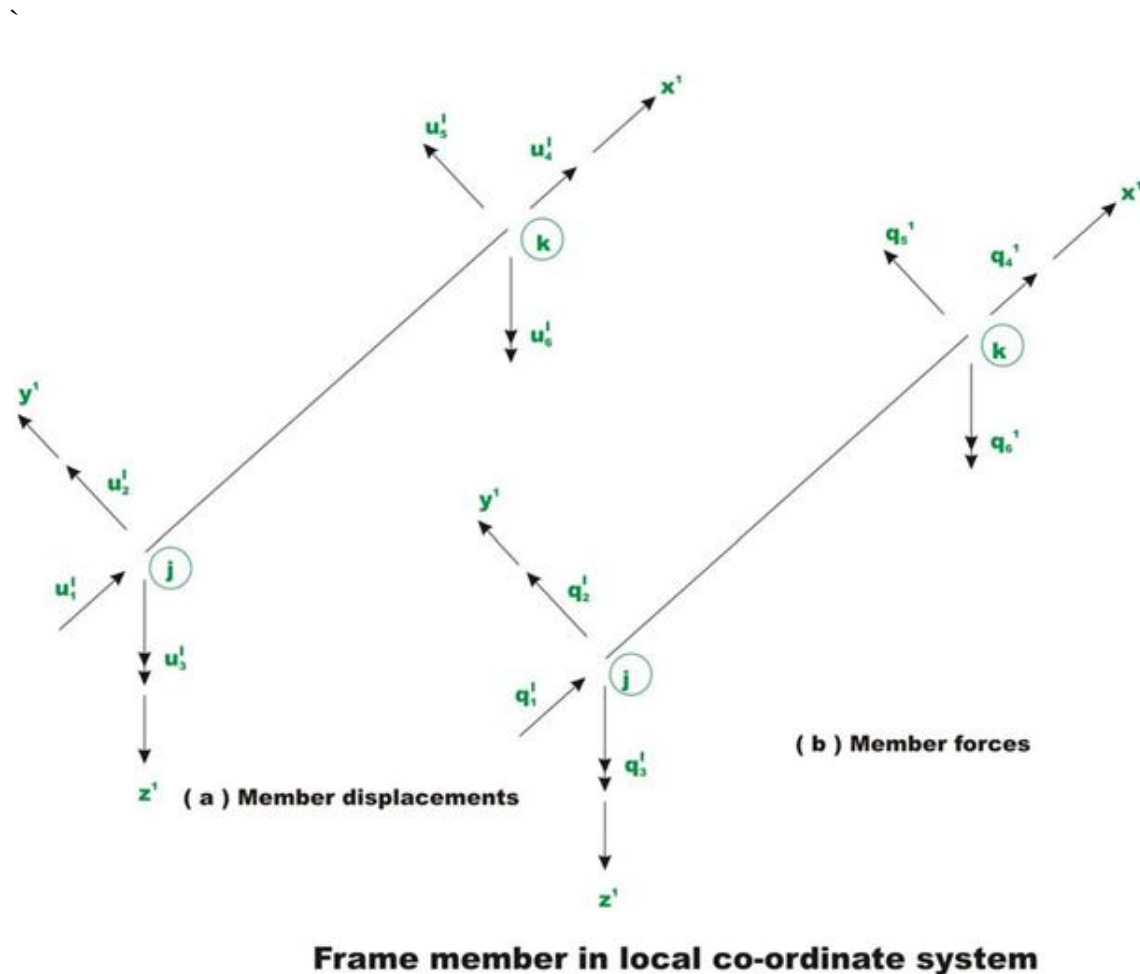


Fig.1

From the above Fig. 1a we can see that the frame members have six (6) degrees of freedom.

Fig. 1b shows the member being subjected to forces at nodes j and k. By merging stiffness matrix for axial effects and the stiffness matrix for flexural effects into a single matrix, the stiffness matrix at the co-ordinates of each node of the frame member is obtained. Thus, the local co-ordinate axes for the plane frame can be represented by the load-displacement relation as shown in Fig a, b as, (eq 1a)

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI_2}{L^3} & \frac{6EI_2}{L^2} & 0 & -\frac{12EI_2}{L^3} & \frac{6EI_2}{L^2} \\ 0 & \frac{6EI_2}{L^2} & \frac{4EI_2}{L} & 0 & -\frac{6EI_2}{L^2} & \frac{2EI_2}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI_2}{L^3} & -\frac{6EI_2}{L^2} & 0 & \frac{12EI_2}{L^3} & -\frac{6EI_2}{L^2} \\ 0 & \frac{6EI_2}{L^2} & \frac{2EI_2}{L} & 0 & -\frac{6EI_2}{L^2} & \frac{4EI_2}{L} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{Bmatrix} \quad \text{--- (1a)}$$

This may be compactly written as (eq 1b),

$$\{q'\} = [k'] \{u'\} \quad \text{--- (1b)}$$

where $[k']$ is the member stiffness matrix. Another method of obtaining the member stiffness matrix is calculating resultant restraint actions by applying one at a time unit displacement along each possible displacement degree of freedom. [11]

3.1.3 Transformation from local to global co-ordinate system

3.1.3.a Displacement transformation matrix

In plane frame, the orientation of the members are in various directions. So the stiffness matrix of a particular member in the local co-ordinate system must be transformed to global co-ordinate system. In Fig. 2a and 2b, the orientation of the plane frame member in local

coordinate axes ($x'y'z'$) and in global coordinate axes (xyz) is represented, respectively. j and k indicates the two opposite ends of the plane frame. Assume u_1', u_2', u_3' to be the displacements at end j and u_4', u_5', u_6' to be the displacement at end k , expressed in the local co-ordinate system. Likewise the displacements of the members at ends j and k expressed in the global co-ordinate system are assumed as u_1, u_2, u_3 and u_4, u_5, u_6 . [11]

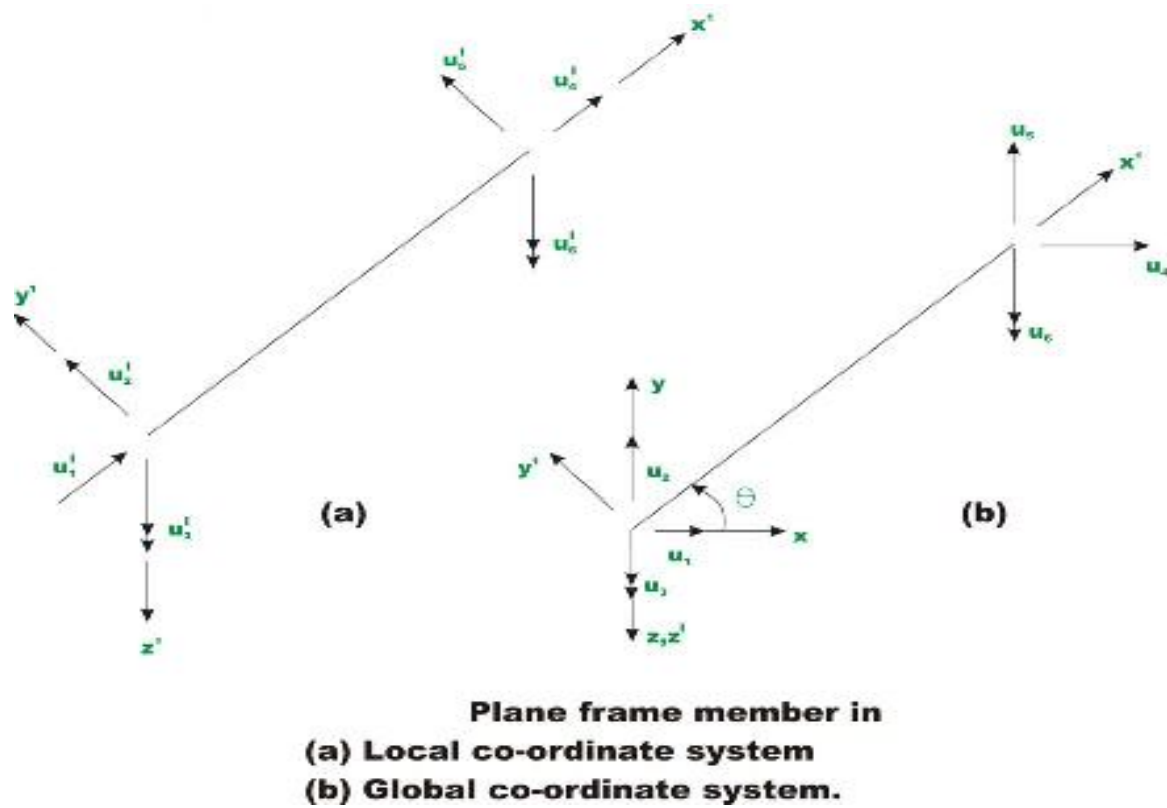


Fig 2.

Taking θ as the angle of inclination of the member to the global x -axis.

From Fig. 2a and b, the following equations are formed,

$$u_1' = u_1 \cos \theta + u_2 \sin \theta \quad \text{--- (2a)}$$

$$-u_1' = u_1 \sin \theta + u_2 \cos \theta \quad \text{--- (2b)}$$

$$u'_3 = u_3 \text{ ————— (2c)}$$

This may be written as (eq 3a)

$$\begin{Bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \\ u'_6 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \text{ ————— (3a)}$$

where $l = \cos \theta$ and $m = \sin \theta$

This may be written in compact form as (eq 3b),

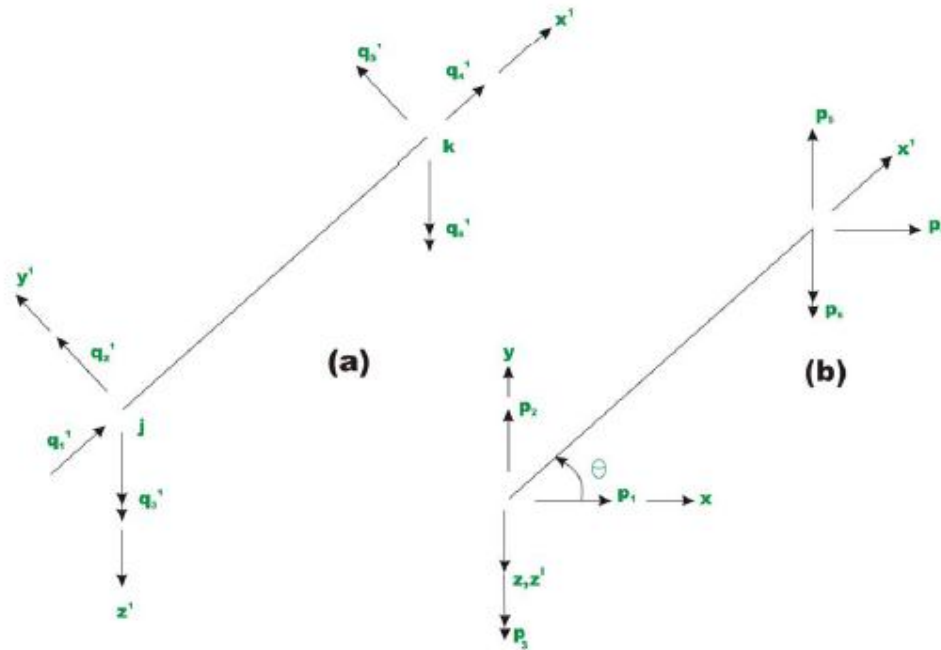
$$\{u'\} = [T] \{u\} \text{ ————— (3b)}$$

In the above equation, the displacement transformation matrix [T] helps in the transformation of six global displacement components to six local displacement components. [11] Assuming the coordinate of node j as (x_1, y_1) and coordinate of node k as (x_2, y_2) then,

$$l = \cos \theta = \frac{x_2 - x_1}{L} \quad \text{and} \quad m = \sin \theta = \frac{y_2 - y_1}{L}$$

$$\text{where } L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ ————— (4)}$$

3.1.3.b Force displacement matrix



Plane frame member in
(a) Local co-ordinate axes and
(b) In global co-ordinate system

Let the forces in member at node j be q_1', q_2', q_3' and at node k be q_4', q_5', q_6' as in Fig. a. These forces are expressed in local co-ordinate system. Similarly, in the global co-ordinate system the member forces at node j are p_1, p_2, p_3 and at node k are p_4, p_5, p_6 , shown in Fig. b. Now from Fig a and b, we get (eq 5a, 5b, 5c respectively), [11]

$$p_1 = \dots \dots \dots (5a)$$

$$p_2 = \dots \dots \dots (5b)$$

$$p_3 = q_3' \quad \text{————— (5c)}$$

Thus the relation between forces in global coordinate system and forces in local coordinate system is expressed by (eq 6a)

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1' \\ q_2' \\ q_3' \\ q_4' \\ q_5' \\ q_6' \end{Bmatrix} \quad \text{————— (6a)}$$

where $l = \cos\theta$ and $m = \sin\theta$

This may be compactly written as (eq 6b),

$$\{p\} = [T]^T \{q'\} \quad \text{————— (6b)}$$

3.1.3.c Member global stiffness matrix

From equation (1b),

$$\{q'\} = [k'] \{u'\}$$

Substituting the above value of $\{q'\}$ in equation (6b) we get,

$$\{p\} = [T]^T [k'] \{u'\} \quad \text{————— (7)}$$

Using equation (3b), the above equation may be rewritten as

$$\{p\} = [T]^T [k'] [T] \{u\} \quad \text{————— (8)}$$

$$\{p\} = [k] \{u\} \quad \text{————— (9)}$$

The equation (9) represents the member load-displacement relation in global coordinate system. The global member stiffness matrix $[k]$ is given by,

$$[k] = [T]^T [k'] [T]$$

Following the similar procedure carried out for truss, member stiffness matrices are assembled, after transformation.. Finally the global load-displacement equation is written as in the case of continuous beam. [11]

3.2 FORMATION OF CONSISTENT MASS MATRIX

3.2.1 Member Mass Matrix

By taking the flexural effects of the members into consideration and by using the consistent mass method, the mass influence coefficient for axial effects of a beam element is found out. Combining the mass matrix for flexural effects with the matrix for axial effects we obtain the consistent mass matrix for a uniform element of a plane frame in reference to the modal coordinates. [11]

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} 140 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 0 & 0 & 0 & 0 \\ 0 & 22L & 4L^2 & 0 & 0 & 0 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & 0 \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}$$

or in condensed notation,

$$\{P\} = [M_C] \{\ddot{\delta}\}$$

in which $[M_C]$ is the consistent mass matrix for the element of a plane frame.

3.2.2 Transformation from local to global co-ordinate system

Repeating the procedure of transformation as applied to the stiffness matrix for the consistent mass matrix, we obtain in a similar manner [11]

$$\{\bar{P}\} = [\bar{M}] \{\bar{\delta}\}$$

in which

$$\{\bar{M}\} = [T]^T [K] [T]$$

3.3 STIFFNESS EQUATION

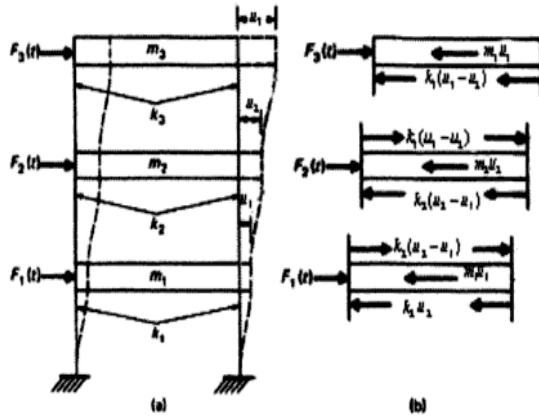
We assume a structure where at the floor levels of the horizontal section, there isn't any rotation. Since the deflected structure behaves like that of a cantilever beam, there are some assumptions that must be taken into consideration:

- 1) concentration of the total mass of the structure at floor levels
- 2) as compared to the columns, the girders on the floors are infinitely rigid
- 3) the axial forces acting on the column have no influence on the structural deformation

The first assumption transforms a multiple-degree-of-freedom system to a structure with degrees equal to the lumped masses at the level of the floors.

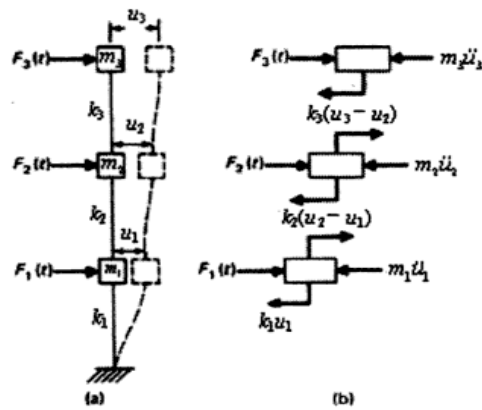
The second assumption indicates that there is no rotation at the joints between girders and columns, i.e. they are fixed against rotation.

The third assumption shows that during motion, rigid girders will be at horizontal position.[9]



(a) Single-bay model representation of a shear building.
(b) Free body diagram

A building with number of bays is expressed in terms of a single bay, i.e. it is idealized into a single column. The idealized structure has mass concentrated at the levels of the floors with only horizontal displacement.



(a) Single-column model representation of a shear building.
(b) Free body diagram.

The stiffness coefficient k_i between any two successive masses is the force required to produce a relative unit displacement of the two adjacent floor levels.[9]

When two ends are fixed against rotation, k_i is expressed as,

$$k = \frac{12EI}{L^3} \text{ -----(1.a)}$$

When one fixed and other pinned,

$$k = \frac{3EI}{L^3} \text{ -----(1.b)}$$

where E is the modulus of elasticity

I is the moment of inertia of cross-section

L is the height of the storey

By equating the sum of all the forces imposed on each mass, the following equations of a three storey structure are obtained,

$$m_1 \ddot{y}_1 + k_1 y_1 - k_2 (y_2 - y_1) - F_1(t) = 0$$

$$m_2 \ddot{y}_2 + k_2 (y_2 - y_1) - k_3 (y_3 - y_2) - F_2(t) = 0$$

$$m_3 \ddot{y}_3 + k_3 (y_3 - y_2) - F_3(t) = 0 \text{ -----(2)}$$

When expressed in matrix form,

$$[M] \{\ddot{y}\} + [K] \{y\} = [F] \text{ -----(3)}$$

$$\text{where } [M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \text{ called the mass matrix}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \text{ called the stiffness matrix}$$

$$\{y\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \text{ called displacement vectors}$$

$$\{\ddot{y}\} = \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} \text{ called acceleration vectors}$$

$$\{F\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix} \text{ called force vectors [9]}$$

3.4 CALCULATION OF EIGEN VALUE AND EIGEN VECTOR

The structure is not excited externally in free vibration mode i.e. no force or support motion acts on it. So, under the condition of free motion, dynamic analysis can be carried out and the important properties like natural frequencies and mode shapes corresponding to the natural frequency can be obtained.

Natural frequencies and normal modes

Since free vibration mode is considered, the structure is not under the influence of any external force. Hence, the force vector in stiffness equations or flexibility equations is taken as zero. [9]

By taking the above condition into consideration, the stiffness equation can be represented as,

$$[M] \{\ddot{y}\} + [K] \{y\} = 0 \text{ -----(1)}$$

The solution of the above equation for undamped structure is in the form,

$$y_i = a_i \sin(\omega t - a) \quad i = 1, 2, 3, \dots, n$$

when represented as vector,

$$\{y\} = \{a\} \sin(\omega t - a) \text{ -----(2)}$$

where a_i is the amplitude of motion of i th coordinate and n is the number of degrees of freedom. Substituting eq. 2 in eq. 1, we get

$$-\omega^2[M] \{a\} \sin(\omega t - a) + [K] \{a\} \sin(\omega t - a) = 0$$

$$\text{or } [[K] - \omega^2[M]] \{a\} = \{0\} \text{-----}(3)$$

The mathematical problem for the formulation of the above equation is called eigenproblem.

As the right hand side of the equation is equal to zero, it can be considered as a set of n number of homogeneous linear equations with n unknown displacements a_i and ω^2 as the unknown parameter. [9]

Amplitude 'a' cannot be zero and so the solution to the eq (3) can be found by considering the determinant of $\{a\}$ to be zero. Its solution is non-trivial. So,

$$[[K] - \omega^2[M]] = 0 \text{-----}(4)$$

The polynomial equation of degree n in ω^2 obtained is called the characteristic equation. The values that satisfy eq (4) can be substituted in eq (3) to obtain the amplitudes in terms of arbitrary constant. [9]

The flexibility equation for free motion is ,

$$\{y\} + [f] [M] \{\ddot{y}\} = \{0\} \text{-----}(5)$$

Assuming the motion to be harmonic, i.e. $\{y\} = \{a\} \sin(\omega t - a)$ and substituting it in eq (5), we get

$$\{a\} = \omega^2 [f] [M] \{a\} \text{-----}(6)$$

$$\text{if } [D] = [f] [M] \text{-----}(7)$$

then,

$$1/\omega^2 \{a\} = [D] \{a\} \text{ -----(8)}$$

$$\text{or } [[D] -1/\omega^2[I]] \{a\} = 0 \text{ -----(9)}$$

Eq (9) results in a trivial solution. The coefficient matrix $\{a\}$ not being zero, its determinant is equated to zero. Thus ,

$$[[D] -1/\omega^2[I]] = 0 \text{ -----(10)}$$

The polynomial equation formulated is of degree n in $1/\omega^2$ and is called the characteristic equation for flexibility formulation. Thus we can calculate the amplitude a_i by substituting the n value obtained from eq (10) into eq (9). [9]

3.5 Orthogonality property of normal modes

Many mechanical and structural systems possessing multiple coordinate systems is needed to describe its motion and vibration sufficiently. Thus a multiple degree of freedom (MDoF) model is formed. The orthogonality property helps in analysing such MDoF systems dynamically.

As already discussed, the equation of motion in free vibration is,

$$[K] \{a\} = \omega^2 [M] \{a\} \text{ -----(11)}$$

Since we are dealing with multiple-degree-of-freedom, let us consider a two-degree-of-freedom system initially. The equation(11) can be rewritten as,

$$(k_1 + k_2) a_1 - k_2 a_2 = m_1 \omega^2 a_1$$

$$-k_2 a_1 + k_2 a_2 = m_2 \omega^2 a_2 \text{ -----(12)}$$

The above equations (12) can be interpreted statically as the forces acting on the system are of the magnitude $m_1 \omega^2 a_1$ and $m_2 \omega^2 a_2$ that are applied to masses m_1 and m_2 , respectively.

The forces acting on either of the two modes result in static deflections. These static deflections help in forming the modal shapes of the member concerned. [9]

The problem with static interpretation can be solved by using the general static theory of linear structures. Betti's theorem states that : For a structure acted upon by two systems of loads and corresponding displacements, the work done by the first system of loads through the displacements of the second system is equal to the work done by the second system of loads undergoing the displacements produced by the first load system.

For two-degree-of-freedom, there are two systems of loading and displacements.

System 1

Forces	$\omega_1^2 a_{11} m_1$	$\omega_1^2 a_{21} m_2$
Displacements	a_{11}	a_{21}

System 2

Forces	$\omega_2^2 a_{12} m_1$	$\omega_2^2 a_{22} m_2$
Displacements	a_{12}	a_{22}

By applying Betti's theorem, we get

$$\omega_1^2 a_{11} m_1 a_{12} + \omega_1^2 a_{21} m_2 a_{22} = \omega_2^2 a_{12} m_1 a_{11} + \omega_2^2 a_{22} m_2 a_{21}$$

$$(\omega_1^2 - \omega_2^2) (a_{11}m_1 a_{12} + a_{22}m_2 a_{21}) = 0 \text{ -----(13)}$$

If $\omega_1 \neq \omega_2$,

$$a_{11}m_1 a_{12} + a_{22}m_2 a_{21} = 0$$

The above equation obtained is called the orthogonality relation. This relation lies between mode shapes of a two-degree-of-freedom system. Similarly, for n-degree-of-freedom system, the relation of orthogonality between any two modes i and j is expressed as

$$\sum_{k=1}^m (m_k)(a_{ki})(a_{kj}) = 0, \quad \text{for } i \neq j \text{ -----(14)}$$

The mass matrix is diagonal.

For n-degree-of-freedom system,

$$\{a_i\}^T [M] \{a_j\} = 0, \quad \text{for } i \neq j$$

where [M] is the mass matrix of the whole system and $\{a_i\}$ and $\{a_j\}$ are the modal vectors.

The amplitudes of the vibration that are obtained in normal mode are not the exact values. The normalized component is expressed as

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\{a_j\}^T [M] \{a_j\}}} \text{ -----(15)}$$

For a diagonal mass matrix system, the above equation can be modified into

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^n (m_k)(a_{kj}^2)}} \text{ -----(16)}$$

We have previously expressed the free modal vibrations of a multiple-degree-of-freedom devoid of any external excitation. Now we will express the modal vibrations for a forced motion of the same system. [9]

The equations of motion for a two-degree-of-freedom system are,

$$\begin{aligned} m_1 \ddot{y}_1 + (k_1 + k_2)y_1 - k_2 y_2 &= F_1(t) \\ m_2 \ddot{y}_2 - k_2 y_1 + k_2 y_2 &= F_2(t) \end{aligned} \quad \text{-----(17)}$$

This coupled equations are required to uncouple such that there is only one unknown function in each equation. The solutions are first expressed in terms of normal modes which are multiplied by factors such that there is contribution of each mode. These factors are general functions designated as $z_i(t)$.

$$\begin{aligned} y_1(t) &= a_{11}z_1(t) + a_{12}z_2(t) \\ y_2(t) &= a_{21}z_1(t) + a_{22}z_2(t) \end{aligned} \quad \text{-----(18)}$$

Substituting (18) into (17), we get

$$\begin{aligned} m_2 a_{11} \ddot{z}_1 - (k_1 + k_2) a_{11} z_1 - k_2 a_{21} z_1 + m_1 a_{12} \ddot{z}_2 + (k_1 + k_2) a_{12} z_2 - k_2 a_{22} z_2 &= F_1(t) \\ m_2 a_{21} \ddot{z}_1 - k_2 a_{11} z_1 + k_2 a_{21} z_1 + m_2 a_{22} \ddot{z}_2 - k_2 a_{12} z_2 + k_2 a_{22} z_2 &= F_2(t) \end{aligned} \quad \text{-----(19)}$$

Here we use the orthogonality relation in which the first equation is multiplied by a_{11} and the second equation by a_{21} . [9]

After simplifying we obtain,

$$(m_1 a_{11}^2 + m_2 a_{21}^2) \ddot{z}_1 + \omega_1^2 (m_1 a_{11}^2 + m_2 a_{21}^2) z_1 = a_{11} F_1(t) + a_{21} F_2(t)$$

Likewise, first equation is multiplied by a_{12} and the second by a_{22} , we get

$$(m_1 a_{12}^2 + m_2 a_{22}^2) \ddot{z}_2 + \omega^2 (m_1 a_{12}^2 + m_2 a_{22}^2) z_2 = a_{12} F_1(t) + a_{22} F_2(t) \quad \text{-----}(20)$$

The above equations interpret that force that is capable of exciting a mode is the work done by the external force causing displacement by the modal shape in question. Each of the equations can be rewritten in the form of single-degree-of-freedom. [9]

$$M_1 \ddot{z}_1 + K_1 z_1 = P_1(t)$$

$$M_2 \ddot{z}_2 + K_2 z_2 = P_2(t) \quad \text{-----}(21)$$

where

$$M_1 = m_1 a_{11}^2 + m_2 a_{21}^2$$

$$M_2 = m_1 a_{12}^2 + m_2 a_{22}^2$$

$$K_1 = \omega_1^2 M_1$$

$$K_2 = \omega_2^2 M_2$$

$$P_1(t) = a_{11} F_1(t) + a_{21} F_2(t)$$

$$P_2(t) = a_{12} F_1(t) + a_{22} F_2(t)$$

Using normalisation ,

$$\ddot{z}_1 + \omega_1^2 z_1 = P_1(t)$$

$$\ddot{z}_2 + \omega_2^2 z_2 = P_2(t) \quad \text{-----}(22)$$

$$\text{where } P_1 = \phi_{11} F_1(t) + \phi_{21} F_2(t)$$

$$P_2 = \phi_{12} F_1(t) + \phi_{22} F_2(t)$$

The solution for these equations can be found by approximate method. Addition of the upper limit of the absolute values of maximum modal contributions gives the upper limit for the maximum response. By adding the absolute values of $z_{1\ max}$ and $z_{2\ max}$ obtained by putting maximum modal responses for z_1 and z_2 in eqs(18), we get

$$y_{1\ max} = |\phi_{11}z_{1\ max}| + |\phi_{12}z_{2\ max}|$$

$$y_{2\ max} = |\phi_{21}z_{1\ max}| + |\phi_{22}z_{2\ max}|$$

This method may overestimate the maximum response. So to obtain a reasonable estimate of maximum response, it is calculated as

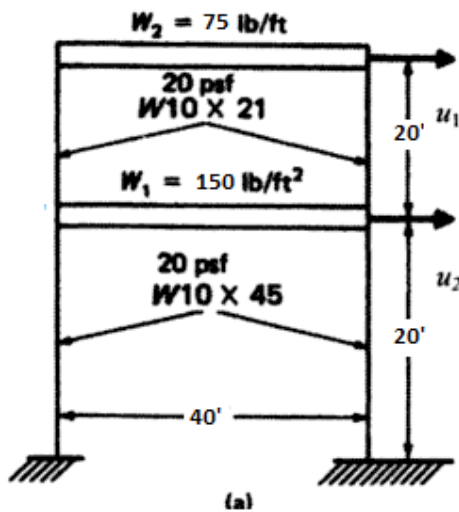
$$y_{1\ max} = \sqrt{(\phi_{11}z_{1\ max})^2 + (\phi_{12}z_{2\ max})^2}$$

$$y_{2\ max} = \sqrt{(\phi_{21}z_{1\ max})^2 + (\phi_{22}z_{2\ max})^2} \quad [9]$$

CHAPTER 4

RESULTS AND COMPARISON

A two-storey steel rigid frame is to be analyzed whose weights of the floors and walls inclusive of the structural weights are indicated in the figure. The frames of the building are 15ft apart. The structural properties along the length of the structure are assumed to be uniform. [9]



Natural Frequencies and Modal Shapes:

According to the lumped mass system, the concentrated weights are the total of floor weight and the weight of tributary walls.

$$W_1 = 150 \times 40 \times 15 + 20 \times 20 \times 15 \times 2 = 102,000 \text{ lb}$$

$$m_1 = 265 \text{ lb} \cdot \text{sec}^2/\text{in}$$

$$W_2 = 75 \times 40 \times 15 + 20 \times 10 \times 15 \times 2 = 51,000 \text{ lb}$$

$$m_2 = 132 \text{ lb} \cdot \text{sec}^2/\text{in}$$

For each storey the stiffness coefficient is calculated as,

$$k = \frac{12E(2I)}{L^3}$$

$$k_1 = \frac{12 \times 30 \times 10^6 \times 248 \times 2}{(20 \times 12)^3} = 13,000 \text{ lb/in}$$

$$k_2 = \frac{12 \times 30 \times 10^6 \times 106.3 \times 2}{(20 \times 12)^3} = 5600 \text{ lb/in}$$

Using dynamic equilibrium on every element of the system which is free from external vibration, the equations of motion are obtained as already discussed which when solved give the natural frequencies as,

$$\omega_1 = 7.542 \text{ rad/sec}$$

$$\omega_2 = 20.334 \text{ rad/sec}$$

in cycles per second

$$f_1 = \omega_1/2\pi = 1.2 \text{ cps}$$

$$f_2 = \omega_2/2\pi = 3.236 \text{ cps}$$

The time periods are,

$$T_1 = 1/f_1 = 0.833 \text{ sec}$$

$$T_2 = 1/f_2 = 0.309 \text{ sec}$$

Substituting $\omega_1 = 7.542 \text{ rad/sec}$ in the matrix equation,

$$3526 a_{11} - 5600 a_{21} = 0$$

$$\frac{a_{21}}{a_{11}} = 0.63$$

By assigning a unit value to one of the amplitudes,

$$a_{11} = 1$$

$$a_{21} = 0.63$$

Likewise putting $\omega_2 = 20.334$ rad/sec , we get the second normal modes,

$$a_{12} = 1$$

$$a_{22} = -6.42$$

Normalized Modal Shapes of Vibration

$$\text{Normalized modes, } \phi_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^n (m_k)(a_{kj}^2)}}$$

By putting the values of amplitudes already calculated and the masses,

$$\sqrt{(265)(1.00)^2 + (132)(1.263)^2} = \sqrt{317.4}$$

$$\sqrt{(265)(1.00)^2 + (132)(-6.42)^2} = \sqrt{5671.7}$$

Normalized modes are,

$$\phi_{11} = \frac{1.000}{\sqrt{317.4}} = 0.056, \quad \phi_{12} = \frac{1.000}{\sqrt{5671.7}} = 0.0132$$

$$\phi_{21} = \frac{1.263}{\sqrt{317.4}} = 0.0354, \quad \phi_{22} = \frac{-6.42}{\sqrt{5671.7}} = -0.0852$$

To satisfy the orthogonality equation,

$$[\emptyset]^T [M] [\emptyset] = [I]$$

$$[\emptyset] = \begin{bmatrix} 0.056 & 0.0132 \\ 0.0354 & -0.0852 \end{bmatrix}$$

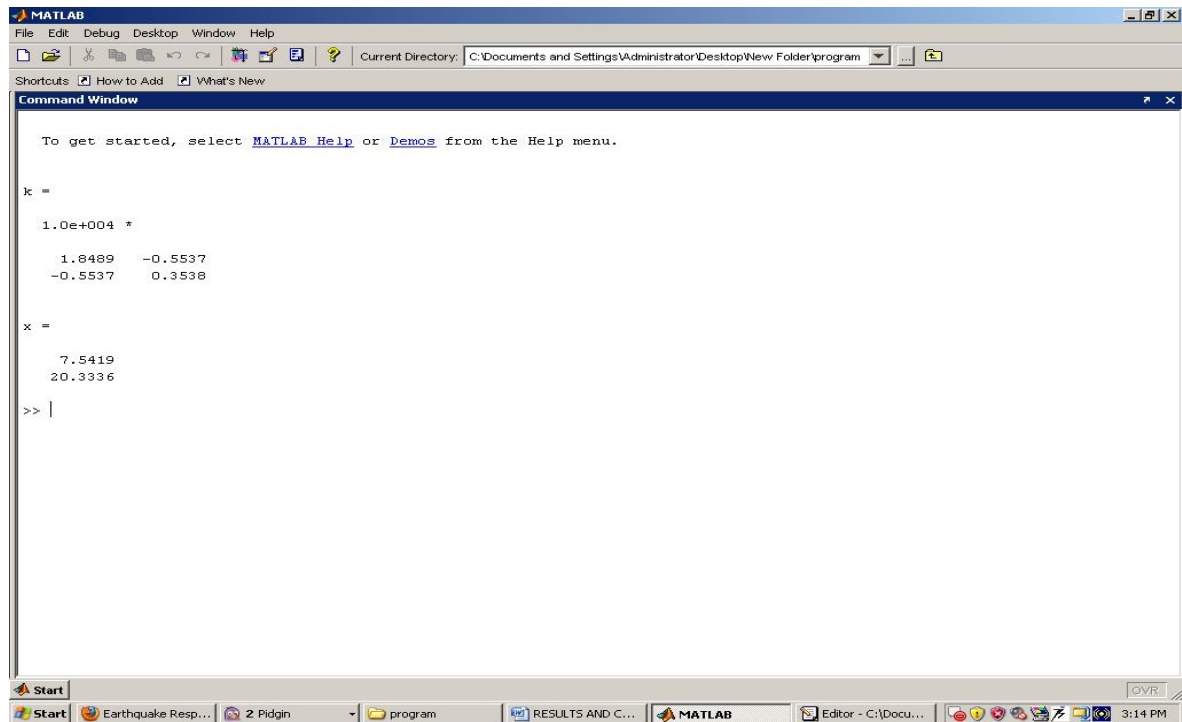
$$= \begin{bmatrix} 0.056 & 0.0354 \\ 0.0132 & -0.0852 \end{bmatrix} \begin{bmatrix} 265 & 0 \\ 0 & 132 \end{bmatrix} \begin{bmatrix} 0.056 & 0.0132 \\ 0.0354 & -0.0852 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

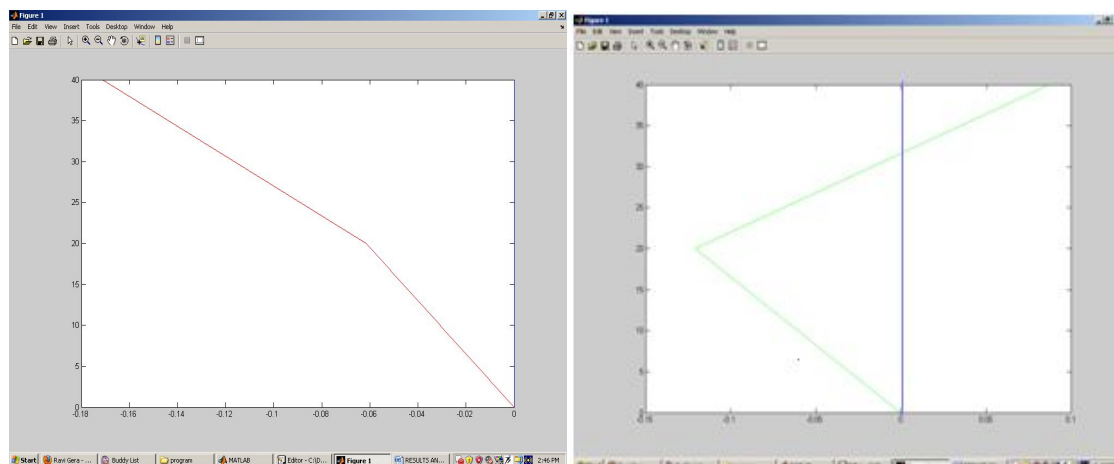
MATLAB Results:

These values are also confirmed by the MATLAB results which gave the following values :

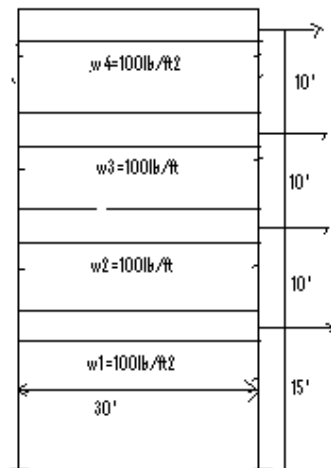
OUTPUT:



Figures below shows the modal shapes for the two modes of the shear building



For four-storey shear building



In the similar way as in the case of two-storey shear building, the natural angular frequencies of the considered system computed by MATLAB were

OUTPUT:

```

MATLAB
File Edit Debug Desktop Window Help
Current Directory: C:\Documents and Settings\Administrator\Desktop\New Folder\program
Shortcuts How to Add What's New

Command Window

To get started, select MATLAB Help or Demos from the Help menu.

enter the value of no of storey 4

K =

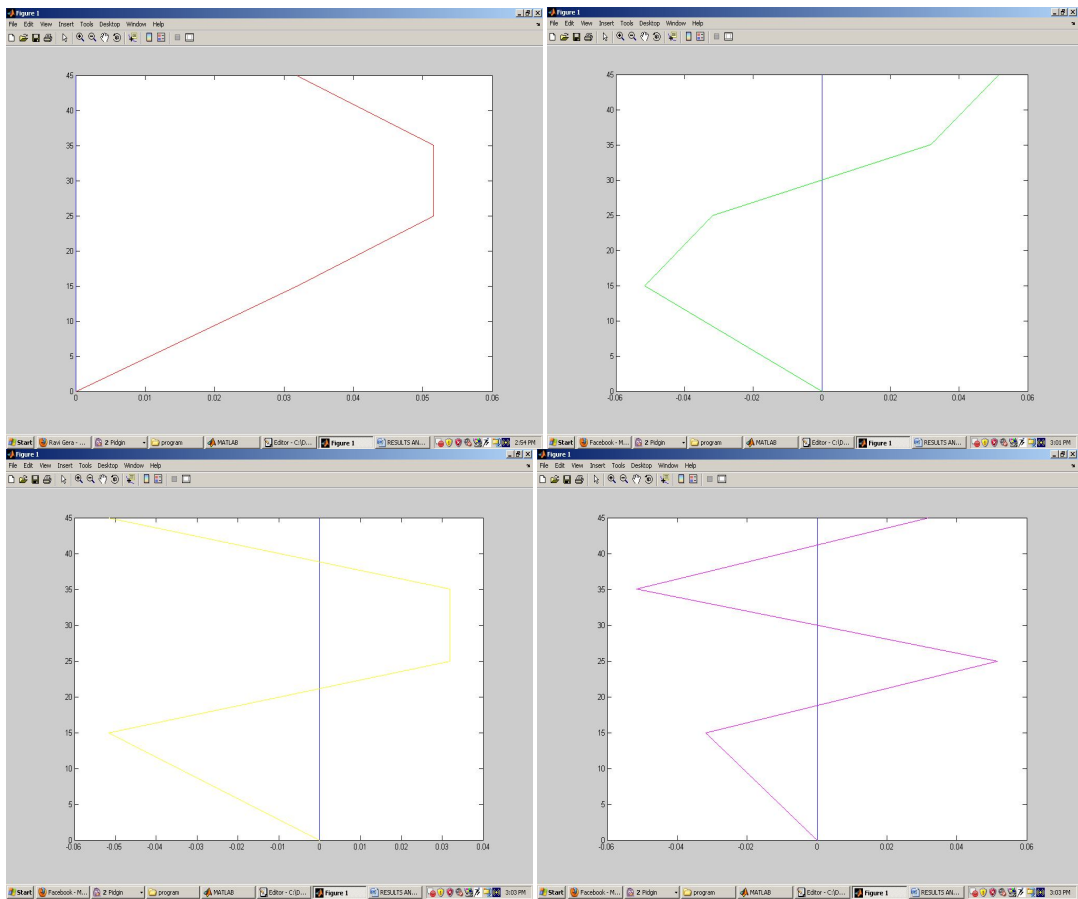
    61400    -30700         0         0
   -30700     61400    -30700         0
         0    -30700     61400    -30700
         0         0    -30700     30700

Y =

    5.2180
   15.0245
   23.0189
   28.2368

>> |
  
```


Figures below shows the modal shapes for the four modes of the shear building



CONCLUSION:

After obtaining the natural frequencies and the eigen vectors for two-storey shear building manually, we compared it with the results of the MATLAB code and the natural frequencies in both case were found out to be 7.542 rad/sec and 20.334 rad/sec.

After getting the same results, we created a generalized MATLAB code for n-storey shear building and obtained the natural frequencies for four-storey building to be 5.218 rad/sec, 15.0245 rad/sec, 23.0189 rad/sec and 28.236 rad/sec.

With the same MATLAB code we obtained the mode shape for each degree of freedom for specific natural frequency already calculated.

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